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New subclasses of analytic functions defined by derivative operator

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Abstract

in the present paper, we introduce and investigate the classes of analytic functions $S^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $Q^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ in the open unit disk $U=\{z\in £ : |z|<1\}$. Some properties such as coefficient estimate, Growth and distortion theorem, extreme pointes for functions $f(z)\in T_n$ will be obtained.

Keywords: Analytic functions, derivative operator, negative coefficient.

* Introduction

Let A denote a class of all analytic functions of the form

1- $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$

which are analytic in the open unit disk $U=\{z\in £ : |z|<1\}$ in the complex plane $£$.

Let A_n denote the class of functions $f(z)$ the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k,\quad (n\in ¥=\{1,2,...\}),$$

which called analytic functions with the negative coefficient in the open unit disk U.

Let T_n denote the subclass of A_n of the form

2- $f(z)=z-\sum_{k=n+1}^{\infty}|a_k|z^k,\quad (n\in ¥=\{1,2,...\})$.

Next, we defined $(n-\varepsilon)$-neighborhood for the functions belonging to the class A_n and also the identity function.

*** Definition 1.1**

Following [8] and [13], for $f \in T_n$ and $\varepsilon \geq 0$, we define the $(n-\varepsilon)$ -neighborhood of $f(z)$ by

3- $\aleph_{n,\varepsilon}(f) = \left\{ f \in T_n : g(z) = z - \sum_{k=n+1}^{\infty} |b_k| z^k \ \ and \ \ \sum_{k=n+1}^{\infty} k\,|a_k - b_k| \leq \varepsilon \right\}$.

In particular, for the identity function $e(z) = z$, we have

4-
$$\aleph_{n,\varepsilon}(e) = \left\{ f \in T_n : g(z) = z - \sum_{k=n+1}^{\infty} |b_k| z^k \ \ and \ \ \sum_{k=n+1}^{\infty} k\,|b_k| \leq \varepsilon \right\}$$
.

Given two functions $f, g \in A, f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$, their convolution or (Hadamard product) $f(z) * g(z)$ is defined by $f(z) * g(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k \ \ (z \in U)$.

And for several functions $f_1(z), f_2(z), \ldots f_m(z) \in A$

$$f_1(z) * f_2(z) * \ldots * f_m = z + \sum_{k=n+1}^{\infty} (a_{1k} a_{1k} \ldots a_{mk}) z^k \ \ (z \in U).$$

We will use the Hadamard product of 1 -th order to define generalized derivative operator.

Ramadn and Darus [11] introduced the operator $I_{\alpha,\beta,\lambda,\delta}^{1,\mu} : A \to A$ defined by

5-
$$I_{\alpha,\beta,\lambda,\delta}^{1,\mu} = z + \sum_{k=2}^{\infty} \frac{(\mu+1)_{k-1}}{\left[(\lambda-\delta)(\beta-\alpha)(k-1)+1\right]^{1} (k-1)!} a_k z^k, \ \ (z \in U)$$
,

$\alpha, \beta, \lambda, \delta, \sigma \geq 0, \lambda > \delta, \beta > \alpha, \gamma > \sigma$,

$1 \in \yen_0 = \yen \cup \{0\}, 1, \rho \in \yen_0$ and $\mu > -1$.

Note that, Pochhammer symbol or (Apple`s symbol) defined by

$$(\mu)_n = \begin{cases} 1, & n = 0 \\ \mu(\mu+1)\ldots(\mu+n-1), & n = 1, 2, \ldots \end{cases}$$

Now, in order to derive our new generalized derivative operator, we assume that:

$$\Psi(z) = \frac{(\beta-\alpha)z}{(1-z)^2} - \frac{(\beta-\alpha)z}{1-z} + \frac{z}{1-z}, \ \beta > \alpha \text{ and } \beta, \alpha \geq 0.$$

Then we obtain the function

$$F_{(1-1)} = \underbrace{\Psi(z) * \Psi(z) * \ldots * \Psi(z)}_{(1-1)-times} * \underbrace{\frac{z}{(1-z)^2} * \ldots * \frac{z}{(1-z)^2}}_{\rho(1-1)-times}$$

6-
$$\Im_{\rho}^{1}(\alpha,\beta) = z + \sum_{k=2}^{\infty} \left[(\beta-\alpha)k^{\rho}(k-1) + k^{\rho}\right]^{1-1} z^k, \ \ (z \in U)$$

where $\alpha, \beta, \lambda, \delta \geq 0, \lambda > \delta, \beta > \alpha$ and $1, \rho \in \yen_0$.

Using Ramadan and Darus derivative operator given by (1.5), and the series given by (1.6), we define our new generalized derivative operator as follows:

*** Definition1.1**

The new operator $\Im_{\mu,\lambda,\delta}^{1,\rho}(\alpha,\beta,\gamma,\sigma) : A \to A$, for $f \in A$, is defined by

7- $\Im_{\mu,\lambda,\delta}^{k,\rho}(\alpha,\beta,\gamma,\sigma) f(z) = \Im_{\rho}^{1}(\alpha,\beta) * I_{\alpha,\beta,\lambda,\delta}^{k,\mu}$

,

where $\alpha, \beta, \lambda, \delta \geq 0, \lambda > \delta, \beta > \alpha$ and $1, \rho \in \yen_0$.

If $f \in A$ is given by (1.1), then from (1.8), we find that

8-
$$\Im_{\mu,\lambda,\delta}^{1,\rho}(\alpha,\beta,\gamma,\sigma) f(z) = z + \sum_{k=2}^{\infty} \frac{\left[(\beta-\alpha)k^{\rho}(k-1) + k^{\rho}\right]^{1-1} (\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1} (k-1)!} a_k z^k, (z \in U),$$

where $\alpha, \beta, \lambda, \delta, \sigma \geq 0, \lambda > \delta, \beta > \alpha, \gamma > \sigma$, $1, \rho \in \yen_0$ and $\mu > -1$.

*** Remark 1**

Special cases of this new operator include:

1- Al-Aboudi derivative operator [2] in the case $\Im_{0,1,0}^{k+1,0}(0,1,0,0)=\Im_{0,0,0}^{k+1,0}(0,1,1,0)=D_k^n$

2- Rusheweyh derivative operator [12] in the case $\Im_{\mu,1,0}^{1,0}(0,1,0,0)=\Im_{\mu,0,0}^{1,0}(0,1,1,0)=R^{\mu}$

3- The generalized Al-Abbadi and Darus derivative operator [1] in the case $\Im_{\mu,\lambda,0}^{k,0}(0,\beta,1,0)=\Im_{\mu,1,0}^{k,0}(0,\beta,\gamma,0)=\mu_{\lambda_1,\lambda_2}^{n,m}$

4- The generalized Darus and Ibrahim derivative operator [6] in the case $\Im_{\mu,1,0}^{k+1,\rho}(0,1,0,0)=\Im_{\mu,0,0}^{k+1,\rho}(0,1,1,0)=D_{\lambda,\delta}^{k,\alpha}$

5- The generalized Al-Shaqsi and Darus derivative operator [4] in the case $\Im_{\mu,1,0}^{k+1,0}(0,1,0,0)=\Im_{\mu,0,0}^{k+1,0}(0,1,1,0)=D_{\lambda}^{m}$.

By using the same method above, we can write the following equality for the function $f(z)$ belonging to the class T_n.

If $f(z)$ given by (1.2), we define the new derivative operator:

9-

$$\Im_{\mu,\lambda,\delta}^{1,\rho}(\alpha,\beta,\gamma,\sigma)f(z)=z-\sum_{k=n+1}^{\infty}\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k|z^k, \quad (z\in U)$$

where $\alpha,\beta,\lambda,\delta,\sigma\geq 0,\lambda>\delta,\beta>\alpha,\gamma>\sigma$, $1,\rho\in ¥_0$ and $\mu>-1$.

A function $f(z)\in A_n$ is said to be a starlike function of complex order ζ or $f(z)\in S^*(\zeta)$ if and only if $f(z)/z\neq 0$ and

$$\Re\left\{1+\frac{1}{\zeta}\left(\frac{zf(z)'}{f(z)}\right)-1\right\}>0, \quad z\in U,\ \zeta\in £\setminus\{0\}.$$

A function $f(z)\in A_n$ is said to be a convex function of complex order ζ or $f(z)\in C^*(\zeta)$ if and only if $f'(z)\neq 0$ and

$$\Re\left\{1+\frac{1}{\zeta}\frac{zf''(z)}{f'(z)}\right\}>0, \quad z\in U,\ \zeta\in £\setminus\{0\}$$

The class $S^*(\zeta)$ was introduced by Naser and Aouf [9]. And the class $C^*(\zeta)$ was introduced by Wiatrowsky [14] and considered by Naser And Aouf[].

Finally, A function $f(z)\in A_n$ is said to be a close-to-convex function of complex order ζ or $f(z)\in K(\zeta)$ if and only if $g(z)/z\neq 0$ and

$$\Re\left\{1+\frac{1}{\zeta}\left(\frac{zf(z)'}{g(z)}\right)-1\right\}>0, \quad z\in U,\ \zeta\in £\setminus\{0\},$$

For some starlike function $g(z)$. This class was introduced by Al-Amiri and Frenando [3].

Now, we define new classes of analytic functions as follows

*** Definition 1.2**

Let $f\in A_n$ Then $f\in S_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ if and only if

10-

$$\Re\left\{\frac{1}{\zeta}\left(\frac{z\left[\Im_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma)f(z)\right]'}{\Im_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma)f(z)}\right)-1\right\}>\eta \quad \left(0\leq\eta<1,\zeta\in £\setminus\{0\}\right),$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\geq 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $,1,\rho\in ¥_0,\mu>-1$ and

$\Im_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma)f(z)/z\neq 0$

*** Definition 1.3**

Let $f \in A_n$ Then $f \in Q^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ if and only if

11-

$$\Re\left\{\frac{1}{\zeta}\left(\Im^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right)'-1\right\}>\eta \quad (0\le\eta<1, \zeta\in £ \setminus\{0\})$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ and $1,\rho\in ¥_0,\mu>-1.$

Further, define the classes $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ by

$$TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)=S^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)\cap T_n$$

and

$$TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)=Q^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)\cap T_n$$

, it is clear that $S^{1,0}_{0,0,0}(1,1,0,0,1,1,\zeta)\subset S^*(\zeta)$, $S^{1+1,0}_{0,0,0}(0,1,0,0,0,0,\zeta)\subset C(\zeta)$ and $Q^{1,0}_{0,0,0}(1,1,0,0,1,1,\zeta)=K(\zeta)$.

Note that various subclasses of $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ have been studied by many authors using suitable choices of parameters, for example $TQ^{0,0}_{0,0,0}(0,1,0,1,1,1-\alpha,\zeta)=R(\alpha)$, was introduced and studied by Sarangi and Uralegaddi [14]. $TQ^{1+1,0}_{0,0,0}(0,\beta,0,0,n,\eta,\zeta)=R(0,\beta,0,0,n,\eta,\zeta)$, was studied by Altintas et. al. [5] and many others.

The $(n-\varepsilon)$-neighborhood of the classes $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, will be studied in the next section.

2- A set of inclusion relations

The following Lemmas, will be required for our investigation of the inclusion relations involving $\aleph_{n,\varepsilon}(e)$.

*** Lemma 2.1**

Let the function $f \in T_n$ be defined by (1.2), then f in the class $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ if and only if

1-

$$\sum_{k=n+1}^{\infty}\left(k+\eta|\zeta|-1\right)\left\{\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}\right\}|a_k|\le\eta|\zeta|,$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in ¥_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in £\setminus\{0\}$.

*** Proof**

We suppose that $f \in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, then from (2.1) we have

$$\Re\left(\frac{z\left[\Im^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right]'}{\Im^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)}-1\right)>-\eta|\zeta|,$$

or equivalent

$$\Re\left\{\frac{-\sum_{k=n+1}^{\infty}\frac{(k-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k||z^k|}{z-\sum_{k=n+1}^{\infty}\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k||z^k|}\right\}>-\eta|\zeta|,$$

When we take the limit for $z\to 1^-$ through real values. We get the above required condition (2.1).

Conversely, by applying the hypotheses (2.1) and letting $|z|=1$, we find that

$$\left|\frac{z\left[\Im_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma)f(z)\right]'}{\Im_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma)f(z)}-1\right|=$$

$$\left|\frac{\sum_{k=n+1}^{\infty}\frac{(k-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k|z^k}{z-\sum_{k=n+1}^{\infty}\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k|z^k}\right|$$

$$\le\left|\frac{\eta|\zeta|\left\{1-\sum_{k=n+1}^{\infty}\frac{(k-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k|z^k\right\}}{1-\sum_{k=n+1}^{\infty}\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k|z^k}\right|=\eta|\zeta|$$

.

This implies that $f\in TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$.

Corollary2.2 Let the function $f\in T_n$ which defined by (1.2) be in the class $TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$. Then we have

$$|a_k|\le\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{\left(k+\eta|\zeta|-1\right)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}\quad(k\ge n+1)$$

,

where

$\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$

$k,\rho\in$ ¥ $_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in$ £ $\setminus\{0\}$.

Lemma 2.3 Let the function $f\in T_n$ be defined by (1.2), then f in the class $TQ_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ if and only if

2-

$$\sum_{k=n+1}^{\infty}\frac{k\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k|\le\eta|\zeta|$$

,

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$

$k,\rho\in$ ¥ $_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in$ £ $\setminus\{0\}$.

Proof: Same as Lemma 2.1.

First inclusion relation involving $\aleph_{n,\varepsilon}(e)$ is given by the following

Theorem 2.4 Let $f\in T_n$ which defined by (2.1), then $TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)\subset\aleph_{n,\varepsilon}(e)$ if

3-

$$\varepsilon:=\frac{\eta|\zeta|(n+1)n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$

$k,\rho\in$ ¥ $_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in$ £ $\setminus\{0\}$.

Proof: For $f\in TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, Lemma 2.1 immediately yields

$$\frac{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}\sum_{k=n+1}^{\infty}|a_k|\le\eta|\zeta|$$

,

so that

$$\sum_{k=n+1}^{\infty}|a_k|\le\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}$$

.

On the other hand, from (2.1) and the above inequality, we have

$$\frac{(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}\sum_{k=n+1}^{\infty}k\,|a_k|$$

$$\le\eta|\zeta|+\frac{\left(1-\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}\sum_{k=n+1}^{\infty}|a_k|$$

$$\le\eta|\zeta|+\frac{\left(1-\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}\times$$

$$\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}$$

$$=$$

$$\eta|\zeta|+\left(1-\eta|\zeta|\right)\frac{\eta|\zeta|}{n+\eta|\zeta|}=\frac{(n+1)\eta|\zeta|}{n+\eta|\zeta|}.$$

Thus

$$\sum_{k=n+1}^{\infty}|a_k|\le\frac{\eta|\zeta|(n+1)n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}},$$

that is

$$\sum_{k=n+1}^{\infty}|a_k|\le\frac{\eta|\zeta|(n+1)n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}=\varepsilon,$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in ¥_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in £\setminus\{0\}$.

Hence by using (1.4), we conclude that $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)\subset\aleph_{n,\varepsilon}(e)$.

Similarly, by applying Lemma 2.3 the following theorem is obtained

Theorem 2.5 Let $f\in T_n$ which defined by (2.1), then $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)\subset\aleph_{n,\varepsilon}(e)$ if

4- $$\varepsilon:=\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}$$

,

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in ¥_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in £\setminus\{0\}$.

Proof: The prove for this theorem is the same as in the Theorem 2.4, so we omit it.

3- Neighborhoods for the classes $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$

In this section, the neighborhoods for the classes $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, will be determinate as follows

A function $f\in T_n$ defined by (2.1) is said to be in the class $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$ if there exists a function $g\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ such that

3- $$\left|\frac{f(z)}{g(z)}-1\right|<1-\tau\quad(0\le\tau<1,\ z\in U).$$

Next, A function $f\in T_n$ defined by (2.1) is said to be in the class $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$ if there exists a function $g\in TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ such that (3.1) holds true.

Theorem 3.1 Let $g\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and

$$\tau=1-\frac{\varepsilon(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{(n+1)\left\{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}-\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}\right\}}$$

then $\aleph_{n,\varepsilon}(g)\subset TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$, where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in ¥_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in £\setminus\{0\}$.

Proof: Suppose that $f\in\aleph_{n,\varepsilon}(g)$. Then we find from (1.3) that

$$\sum_{k=n+1}^{\infty}k\,|a_k-b_k|\le\varepsilon,$$

Implies that

$$\sum_{k=n+1}^{\infty}|a_k-b_k|\le\frac{\varepsilon}{n+1}\qquad(n=1,2,...),$$

since $g\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, we have

$$\sum_{k=n+1}^{\infty}|b_k|\le\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}.$$

Now

$$\left|\frac{f(z)}{g(z)}-1\right|\le\frac{\sum_{k=n+1}^{\infty}k\,|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}$$

$$\le\frac{\varepsilon}{n+1}\times\frac{1}{1-\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}}$$

$$=\frac{\varepsilon\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{(n+1)\left\{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}-\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}\right\}}$$

$=1-\tau$.

This means that $f\in TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$, therefore $\aleph_{n,\varepsilon}(g)\subset TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$. And hence the proof is complete.

To prove the following theorem, we should use the same method of Theorem 3.1.

Theorem 3.1 Let $g\in TQ_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and

$$\tau=1-\frac{\varepsilon(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}}{(n+1)\left\{(\mu+1)_n\left[(n+1)^{\rho}\left(n(\beta-\alpha)+1\right)\right]^{1-1}-\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}\right\}}$$

,

then $\aleph_{n,\varepsilon}(g)\subset TQ_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$, where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in$ ¥$_0$, $\mu>-1$, $0\le\eta<1$ and $\zeta\in$ £ $\setminus\{0\}$.

4- Growth and distortion theorems

In this section a growth and distortion property for function f defined by (1.2) in the classes $TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$ and $TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta,\tau)$, will be given as the following

Theorem 4.1 Let the function f which defined by (1.2) be in the class $TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$. Then for $|z|<1$ we have

$$|z|-\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}\left|z^{n+1}\right|\le\left|f(z)\right|\le$$

$$|z|+\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}\left|z^{n+1}\right|,$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in$ ¥$_0$, $\mu>-1$, $0\le\eta<1$ and $\zeta\in$ £ $\setminus\{0\}$.

Proof: Let $f\in TS_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, then from (2.1) we have

$$\frac{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}{n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}\sum_{k=n+1}^{\infty}|a_k|\le$$

$$\sum_{k=n+1}^{\infty}\left(k+\eta|\zeta|-1\right)\left\{\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}\right\}|a_k|\le\eta|\zeta|$$

.

Hence

$$\left|f(z)\right|=\left|z-\sum_{k=n+1}^{\infty}a_kz^k\right|\le|z|+\sum_{k=n+1}^{\infty}|a_k|\left|z^k\right|,$$

or

$$\left|f(z)\right|\le|z|+\sum_{k=n+1}^{\infty}|a_k|\left|z^k\right|$$

$$\left|f(z)\right|\le|z|+\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}\left|z^{n+1}\right|.$$

Similarly

$$\left|f(z)\right|=\left|z-\sum_{k=n+1}^{\infty}a_kz^k\right|\ge|z|-\sum_{k=n+1}^{\infty}|a_k|\left|z^k\right|,$$

or

$$\left|f(z)\right|\ge|z|-\sum_{k=n+1}^{\infty}|a_k|\left|z^k\right|.$$

This implies that

$$\left|f(z)\right|\ge|z|-\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{\left(n+\eta|\zeta|\right)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}\left|z^{n+1}\right|.$$

Thus completes the proof.

Theorem 4.2 Let the function f which defined by (1.2) be in the class $TQ_{\mu,\lambda,\delta}^{1,p}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$. Then for $|z|<1$ we have

$$|z|-\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+1)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}\left|z^{n+1}\right|\le\left|f(z)\right|\le$$

$$|z|+\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+1)(\mu+1)_n\left[(n+1)^{\rho}\left((\beta-\alpha)n+1\right)\right]^{1-1}}\left|z^{n+1}\right|,$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\geq 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\leq\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$.

Theorem 4.3 Let the function f which defined by (1.2) be in the class $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$. Then for $|z|<1$ we have

$$|z|-\frac{\eta|\zeta|}{(n+\eta|\zeta|)}|z^{n+1}|\leq\left|\Im^{1,\rho}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right|\leq|z|+\frac{\eta|\zeta|}{(n+\eta|\zeta|)}|z^{n+1}|,$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\geq 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\leq\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$.

Proof: Let $f\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$, then from (1.8) we have

$$(n+\eta|\zeta|)\sum_{k=n+1}^{\infty}\left\{\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}\right\}|a_k|\leq$$

$$\sum_{k=n+1}^{\infty}(k+\eta|\zeta|-1)\left\{\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}\right\}|a_k|\leq\eta|\zeta|$$

.

Hence

$$\left|\Im^{1,\rho}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right|=\left|z-\sum_{k=n+1}^{\infty}\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}a_k z^k\right|$$

,

and

$$\left|\Im^{1,\rho}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right|\geq|z|-\sum_{k=n+1}^{\infty}\frac{\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}|a_k||z^k|,$$

We have

$$\left|\Im^{1,\rho}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right|\geq|z|-\frac{\eta|\zeta|}{(n+\eta|\zeta|)}|z^{n+1}|$$

.

Similarly, we find that

$$\left|\Im^{1,\rho}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right|\leq|z|+\frac{\eta|\zeta|}{(n+\eta|\zeta|)}|z^{n+1}|$$

.

Theorem 4.4 Let the function f which defined by (1.2) be in the class $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$. Then for $|z|<1$ we have

$$|z|-\frac{\eta|\zeta|}{n+1}|z^{n+1}|\leq\left|\Im^{1,\rho}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma)f(z)\right|\leq|z|+\frac{\eta|\zeta|}{n+1}|z^{n+1}|,\quad n=1,2,\ldots$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\geq 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\leq\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$.

Theorem 4.5 Let the hypothesis of the theorem 4.1 be satisfied, then

$$1-\frac{\eta|\zeta|(n+1)n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}((\beta-\alpha)n+1)\right]^{1-1}}|z^n|\leq|f'(z)|\leq$$

$$1+\frac{\eta|\zeta|(n+1)n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(n+\eta|\zeta|)(\mu+1)_n\left[(n+1)^{\rho}((\beta-\alpha)n+1)\right]^{1-1}}|z^n|$$

,

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\geq 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\leq\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$.

Theorem 4.6 Let the hypothesis of theorem 4.2 be satisfied, then

$$1-\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(\mu+1)_n\left[(n+1)^{\rho}((\beta-\alpha)n+1)\right]^{1-1}}|z^n|\leq|f'(z)|\leq$$

$$1+\frac{\eta|\zeta|n!\left[n(\gamma-\sigma)(\lambda-\delta)+1\right]^{1}}{(\mu+1)_n\left[(n+1)^{\rho}((\beta-\alpha)n+1)\right]^{1-1}}|z^n|$$

,

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\geq 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\leq\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$.

5- Extreme points

The extreme points of the classes $TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ and $TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ will be obtained as follows

Theorem 5.1 (i) Let $f_n(z)=z$ and

$$f_k(z)=z-\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}z^k \quad (k\ge n+1),$$

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$. Then $f\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ if and only if it can be expressed in the form $f(z)=\sum_{k=n}^{\infty}\Delta_k f_k(z)$, where $\Delta_k\ge 0$ and $\sum_{k=n}^{\infty}\Delta_k=1$

(ii) Let $f_n(z)=z$ and

$$f_k(z)=z-\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{k\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}z^k \quad (k\ge n+1)$$

,

where $\alpha,\beta,\lambda,\gamma,\delta,\sigma\ge 0,\ \beta>\alpha,\lambda>\delta,\gamma>\sigma,$ $k,\rho\in\mathbb{N}_0,\mu>-1,\ 0\le\eta<1$ and $\zeta\in\mathbb{C}\setminus\{0\}$. Then $f\in TQ^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ if and only if it can be expressed in the form $f(z)=\sum_{k=n}^{\infty}\Delta_k f_k(z)$, where $\Delta_k\ge 0$ and $\sum_{k=n}^{\infty}\Delta_k=1$.

Proof: suppose that

$$f(z)=\sum_{k=n}^{\infty}\Delta_k f_k(z)=\Delta_n f_n(z)+\sum_{k=n+1}^{\infty}\Delta_k f_k(z)$$

$$=\Delta_n z+\sum_{k=n+1}^{\infty}\Delta_k\left[z-\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}z^k\right]$$

$$=\Delta_n z+\sum_{k=n+1}^{\infty}\Delta_k z-\sum_{k=n+1}^{\infty}\Delta_k\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}z^k$$

$$=\left(\sum_{k=n}^{\infty}\Delta_k\right)z-\sum_{k=n+1}^{\infty}\Delta_k\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}z^k$$

$$=z-\sum_{k=n+1}^{\infty}\Delta_k\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}z^k$$

.

Then

$$=\sum_{k=n+1}^{\infty}\Delta_k\left\{\frac{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}\times\right.$$

$$\left.\frac{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}\right\}$$

$$=\sum_{k=n+1}^{\infty}\Delta_k=\sum_{k=n}^{\infty}\Delta_k-\Delta_n=1-\Delta_n\le 1.$$

Thus $f\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$ by the condition (2.1). conversely, suppose that $f\in TS^{1,p}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\sigma,n,\eta,\zeta)$. By using corollary 2.2 we may set

$$\Delta_k=\frac{(k+\eta|\zeta|-1)\left[(\beta-\alpha)k^{\rho}(k-1)+k^{\rho}\right]^{1-1}(\mu+1)_{k-1}}{\eta|\zeta|\left[(\gamma-\sigma)(\lambda-\delta)(k-1)+1\right]^{1}(k-1)!}a_k \quad (k\ge n+1)$$

and $\Delta_n=1-\sum_{k=n+1}^{\infty}\Delta_k$. Then we obtain

$$f(z)=\sum_{k=n}^{\infty}\Delta_k f_k(z).$$

The proof of the part (ii) of Theorem 5.1 is similar to the part (i).

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