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Proving the validity of the Riemann hypothesis using the modulus

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ABSTRACT

In this article, it is proved that the non-trivial zeros of Riemann zeta function must lie on the critical line, known as the Riemann hypothesis.

Keywords Riemann zeta function. Riemann hypothesis, Non-trivial zeros, Critical line, Modulus.

*** Introdution**

1- Riemann zeta function: Riemann zeta function is defined over the complex plane (Riemann [1859])

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad , \quad Re(s) > 1 \qquad (1)$$

Where Re(s) denotes the real part of s. There are several forms can be used for an analytic continuation for any s in $\mathbb{C}$, such as (Riemann [1859]),

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{-(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \qquad (2)$$

2- Zeros of the Riemann zeta function: The trivial zeros of the Riemann zeta function occur at the negative even integers; that is, $\zeta(-2n) = 0\ , n = 1,2,3, \dots ....$ (Riemann [1859])

On the other hand. the non-trivial zeros lie in the critical strip, $0 < R(s) < 1$, and are known to be symmetric about the x-axis and the critical line $Re(s) = \frac{1}{2}$; that is $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$ (Riemann [1859], Borwein et al. [2007])

3- Riemann hypothesis: All the non-trivial zeros of the Riemann zeta function lie on the critical line $(s) = \frac{1}{2}$.

*** Proof**

1- Modulus and conjugate of a complex variable

Points to remember: -

$$\overline{\zeta(s)} = \zeta(\bar{s})$$

If $\zeta(a+ib) = x+iy$ then $\zeta(a-ib) = x-iy$, since $|x+iy| = |x-iy|$, then $|\zeta(a+ib)| = |\zeta(a-ib)|$, also $|a+ib| = |a-ib|$ $(a, b \in \mathbb{R}\setminus\{0\})$, $(x, y \in \mathbb{R})$

From which we get that: -

$|\zeta(a+ib)| = |\zeta(a-ib)|$ iff $|a+ib| = |a-ib|$ (3)

$|\zeta(s)| = |\zeta(\bar{s})| \overset{s}{\Leftrightarrow} |s| = |\bar{s}|$ (4)

Using equation (2), we can write,

$$\zeta(s) = \pi^{\frac{2s-}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s) \quad (5)$$

which is also we can write,

$$\zeta(1-s) = \pi^{\frac{1-2s}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s) \quad (6)$$

Note that from equations (٥),(٦) , we conclude that when $\zeta(s) = 0$ to be $\zeta(1-s) = 0$, and vice versa , that is : -

$$\zeta(s) = 0 \overset{s}{\Leftrightarrow} \zeta(1-s) = 0$$

From which we get that $\zeta(s) = \zeta(1-s) = 0$, meaning that: -

$$\zeta(a+ib) = \zeta(1-a-ib) = 0$$

$$\zeta(a+ib) = \zeta((1-a)-ib) = 0$$

(٧)

From the modulus properties were applied, using the relationship (4), therefore equation (٧) becomes,

$|\zeta(s)| = |\zeta(1-s)| \overset{ssss}{\Longrightarrow} |(s)| = |(1-s)|$

And by squaring all sides, we get ,

$$(|(s)|)^2 = (|(1-s)|)^2$$

$$(|(a+ib)|)^2 = (|((1-a)-ib)|)^2$$

$$(a)^2 + (b)^2 = (1-a)^2 + (-b)^2$$

$$a^2 + b^2 = 1 - 2a + a^2 + b^2$$

$$1 - 2a = 0$$

$$2a = 1$$

$$a = \frac{1}{2}$$

That is, a non-trivial zero must lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

*** Another proof**

Since the non-trivial zeros lie in the critical strip, $0 < \mathrm{Re}(s) < 1$, and are known to be symmetric about the x-axis and the critical line $\mathrm{Re}(s) = \frac{1}{2}$; that is

$$\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$$

(Riemann [1859], Borwein et al. [2007])

From which we conclude that:

$|\zeta(s)| = |\zeta(1-s)| = |\zeta(\bar{s})| = |\zeta(1-\bar{s})| = 0$ (٨)

From the symmetric property of the non-trivial zeros of the zeta function, the symmetric of zeta function about the x-axis, where the modulus properties were applied, using the formula (4), therefore equation (٨) becomes,

$|(s)| = |(1-s)| = |(\bar{s})| = |(1-\bar{s})|$ (٩)

And by squaring all sides, we get,

$$(|(s)|)^2 = (|(1-s)|)^2 = (|(\bar{s})|)^2 = (|(1-\bar{s})|)^2$$

Or equivalently,

$$(|(s)|)^2 + (|(1-s)|)^2 + (|(\bar{s})|)^2 + (|(1-\bar{s})|)^2 = 4(|(s)|)^2$$

Substituting $s = a + ib$ where $(a, b \in \mathbb{R}\backslash\{0\})$

$$(|(a+ib)|)^2 + (|(1-a-ib)|)^2 + (|(a-ib)|)^2 + (|(1-a+ib)|)^2 = 4(|(a+ib)|)^2$$

$$(|(a+ib)|)^2 + (|((1-a)-ib)|)^2 + (|(a-ib)|)^2 + (|((1-a)+ib)|)^2 = 4(|(a+ib)|)^2$$

$$(a)^2 + (b)^2 + (1-a)^2 + (-b)^2 + (a)^2 + (-b)^2 + (1-a)^2 + (b)^2 = 4((a)^2 + (b)^2)$$

$$a^2 + b^2 + 1 - 2a + a^2 + b^2 + a^2 + b^2 + 1 - 2a + a^2 + b^2 = 4a^2 + 4b^2$$

$$4a^2 + 4b^2 - 4a + 2 = 4a^2 + 4b^2$$

$$-4a + 2 = 0$$

$$4a = 2$$

$$a = \frac{2}{4}$$

$$a = \frac{1}{2}$$

That is, a non-trivial zero must lie on the critical line $Re(s) = \frac{1}{2}$.

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